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Gauge-Invariant Perfect-Fluid Robertson–Walker Perturbations

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In the preceding paper, a complete set of basic gauge-invariant variables was defined that uniquely characterizes cosmological perturbations in homogeneous, isotropic, ideal-fluid universe models. The calculations were presented in some detail for the case of a general perfect fluid with two essential thermodynamic variables. Among other things, it was demonstrated that the aforementioned set consists of 17 linearly independent, not identically vanishing gauge-invariant variables. One can think of these basic variables as having two aspects. First, their definitions are such that they provide a unique representation of the physical perturbation. (By way of digression, inspection shows that such perturbations can be regarded as being the elements of a certain quotient space.) Second, any complicated gauge-invariant quantity is obtainable directly from the basic variables through purely algebraic and differential operations. The object here is the systematic derivation of the linear propagation equations governing the evolution of these basic variables. To make clear the relation of the present formalism to a series of standard results in the literature, this paper also points out how general propagation equations can be adapted to situations where the pressure vanishes in the background. Finally, the physical interpretation of basic variables and comparison with other gauge-invariant approaches are briefly presented.

1. INTRODUCTION

The difficulties facing the theory of perturbations in homogeneous and isotropic cosmological models are well known. A formal description is available if one exploits the viewpoint that the direct way to formulate linear (or higher order) perturbation theory for the general-relativistic field equations is to use one-parameter families of exact solutions to these equations (Ehlers,

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1973; Wald, 1984). In a neighborhood of a background solution, one may then be able to effectively construct such families by first selecting out of the nonlinear equations the part linear in the "amplitude" of the perturbation and next restricting attention only to this part. However, the resulting theory is plagued by the problem that the choice of variables to represent the inhomogeneities depends on the gauge chosen. Hence, probably, with gauge-dependent variables there is no possibility of explaining the origin of perturbations, which eventually give rise to galaxies or clusters of galaxies.

One reasonable way out of this impasse is to look for gauge-invariant quantities that code the information we need to discuss density inhomogeneities in an almost-Robertson–Walker universe model. The construction of these quantities was pioneered by Bardeen (1980), who gave the first gauge-invariant treatment of the linearized Einstein field equations for the most general matter-associated perturbations away from a homogeneous and isotropic space-time. Based on the theoretical framework of Hawking (1966), Ellis and Bruni (1989) also devoted considerable efforts to the problem of adapting perturbation theory to the requirements of gauge invariance. However, even if we provide a clear interpretation of how the gauge-invariant quantities are related to density perturbations, there does remain one additional complication which simply cannot be avoided: Due to the gauge freedom in general relativity corresponding to the group of diffeomorphisms of space-time, two perturbations $\delta\mathcal{G}_0$ and $\delta\mathcal{G}_0'$ are equivalent if they differ by the action of an "infinitesimal diffeomorphism" on the background solution (Ehlers, 1973; Banach and Piekarski, 1996a). Strictly speaking, then, the object of most physical interest is not just one perturbation $\delta\mathcal{G}_0$, but a whole equivalence class $[\delta\mathcal{G}_0]$ of all perturbations $\delta\mathcal{G}_0'$ which are equivalent to $\delta\mathcal{G}_0$. Another way of looking at this is to say that the gauge problem is a consequence of the general covariance of Einstein's theory of gravity (Hawking and Ellis, 1973). Therefore the physical perturbations $[\delta\mathcal{G}_0]$ are elements of a certain quotient space, and the gauge-invariant variables should be such as to enable a unified and transparent description of this quotient space.

In a previous paper (Banach and Piekarski, 1996a), we first defined a complete set of basic gauge-invariant variables with a nontrivial geometrical meaning and then proved that the equivalence class $[\delta\mathcal{G}_0]$ is uniquely determined from these basic variables. For nonbarotropic perfect fluids, we have found that the aforementioned set consists of 17 linearly independent, not identically vanishing gauge-invariant quantities. Also, we have shown there that any complicated gauge-invariant quantity can be constructed directly from the basic variables through purely algebraic and differential operations. The object here is the systematic derivation of the linear propagation equations governing the evolution of these basic variables. The calculations are presented in some detail for the case of a general perfect fluid with nonzero pressure, the physics of which is expressed by a suitable equation of state involving two essential thermodynamic variables (Misner *et al.*, 1973). Much of the literature on relativistic hydrodynamics in the late universe examines only the behavior of a perfect fluid for pressure-free flows. Our discussion will not make this restriction, but later in the text we shall point out how linear propagation equations can be adapted to situations where the pressure vanishes in the background. Finally, by giving an explicit interpretation of the Ellis and Bruni variables (Ellis and Bruni, 1989) in terms of our formalism, we shall illustrate the general thesis of Banach and Piekarski (1996a) that any gauge-invariant quantity can be defined directly from the basic variables.

To the best of our knowledge, this paper is also the first analysis which explains why the gauge invariance in itself does not completely resolve the ambiguity of what one means by a physical solution to the linearized field equations. As a matter of fact, in Sections 3.2 and 4.2 we shall prove that some additional arguments are always necessary to eliminate the unphysical solutions of gauge-invariant equations. Here it is perhaps important to mention that these spurious *gauge-invariant* solutions have nothing to do with the "gauge mode" solutions which can be directly annulled by a gauge transformation and thus have a different explanation and status.

For simplicity, this paper will focus upon the issue of deriving linear propagation equations in the case when the background space-time considered is described by a Robertson–Walker metric with flat spatial sections ($k = 0$). It should, however, be stressed that in our approach essentially nothing changes if we allow for nonzero background three-space curvature. Indeed, in Sections 5.2 and 6 of Banach and Piekarski (1996a) we tried to include enough introductory material for three-spaces of positive or negative curvature or flat space ($k = +1, -1, 0$) so that the reader can easily pursue the topic of her or his interest further. Specifically, we have derived there a complete set of basic gauge-invariant variables without imposing any restrictions on the curvature k of a three-space.

We organize our paper in the following way. In Section 2 we give, for the reader's convenience, a short account of perturbation theory in its "naive" formulation, but drop any assumptions about the gauge. In Section 3 we introduce basic gauge-invariant variables and then rewrite the linearized field equations in terms of these variables. Contact with the standard results is made in Section 4, where we show how the linearized field equations look when the pressure vanishes in the background. The physical meaning of basic gauge-invariant variables is discussed in Section 5, in order to find a simple representation of density inhomogeneities in an almost-Robertson–Walker universe model. Section 6 is for discussion, conclusion, and comparison with other gauge-invariant approaches.

One additional word regarding the notation and presentation of this paper. For brevity, without further comment we shall use those symbols and results which either are reasonably standard or appear for the first time in Banach and Piekarski (1996a), and the analysis proceeds in a way similar to that already made familiar.

2. REVIEW OF THE THEORY OF PERTURBATIONS

2.1. Definition of a Background Solution

The equations of relativistic linear perturbation theory are most conveniently derived by linearizing Einstein's equations,

$$R^{\alpha\beta} - \tfrac{1}{2}R^{\nu}{}_{\nu}\, g^{\alpha\beta} = T^{\alpha\beta} \tag{2.1a}$$

and the equation of balance of number density,

$$N^{\alpha}{}_{;\alpha} = 0 \tag{2.1b}$$

about a $k = 0$ Robertson–Walker background solution. The stress-energy tensor is taken to be that of a general perfect fluid,

$$T^{\alpha\beta} = (e + p)u^{\alpha}u^{\beta} + pg^{\alpha\beta} \tag{2.2}$$

where e and p are the energy density and pressure of the fluid and u^{α} is the fluid coordinate peculiar velocity. Here we shall restrict our attention to equations of state of the form

$$e = e(n, T), \qquad p = p(n, T) \tag{2.3}$$

where n is the number density and T is the temperature. Clearly, the number flux vector N^{α} satisfies the property that

$$N^{\alpha} = nu^{\alpha} \tag{2.4}$$

Knowing n, T, u^{α}, and $g^{\alpha\beta}$, one may obtain $T^{\alpha\beta}$ and N^{α} from equations (2.2)–(2.4). Thus the general perfect fluid can be described by giving a number density n, a temperature T, a four-velocity u^{α} normalized by $u^{\alpha}u_{\alpha} = -1$, and a contravariant metric tensor $g^{\alpha\beta}$, which are required to obey equations (2.1a) and (2.1b).

In the case of a $k = 0$ Robertson–Walker geometry, it is not difficult to show that the evolution equations assume the form (Peebles, 1993)

$$3H^2 = e_0 \tag{2.5a}$$

$$-6(\dot{H} + H^2) = e_0 + 3p_0 \tag{2.5b}$$

$$\dot{e}_0 + 3(e_0 + p_0)H = 0 \tag{2.5c}$$

$$\dot{n}_0 + 3n_0H = 0 \tag{2.5d}$$

where H is Hubble's parameter related to the expansion factor R by

$$H := \dot{R}/R \tag{2.6}$$

and where an overdot indicates differentiation with respect to time. In equations (2.5), the objects e_0 and p_0 are defined by

$$e_0 := e(n_0, T_0), \qquad p_0 := p(n_0, T_0) \tag{2.7}$$

As to the meaning of n_0 and T_0, these are, respectively, the background number density and the background temperature. Thus, in this case, the independent dynamical variables are chosen to be $R(t)$, $n_0(t)$, and $T_0(t)$. Equation (2.5c) then becomes

$$\dot{T}_0 = -\frac{3}{e_T}(e_0 - n_0e_M + p_0)H \tag{2.8}$$

where

$$e_M := \partial e_0/\partial n_0, \qquad e_T := \partial e_0/\partial T_0 \tag{2.9}$$

2.2. Gauge-Dependent Perturbations

We use the same notation as in Banach and Piekarski (1996a) and thus we define $\mathcal{G}_\epsilon(x)$ by

$$\mathcal{G}_\epsilon(x) := (g^{\alpha\beta}(\epsilon, x), u^\alpha(\epsilon, x), n(\epsilon, x), T(\epsilon, x)) \tag{2.10}$$

Consider an open interval $I := (-d, d)$ of R, $d > 0$. Let $\{\mathcal{G}_\epsilon; \epsilon \in I\}$ be a one-parameter family of solutions of equations (2.1a) and (2.1b). It may be thought of as a "curve" in the space of solutions passing through the "point" given by

$$\mathcal{G}_0(x) := \mathcal{G}_\epsilon(x)_{|\epsilon=0} \tag{2.11}$$

which we call the background solution. To illustrate the use of perturbation techniques, we consider the case of a $k = 0$ Robertson–Walker background solution; thus we set

$$g^{00}(0, x) = -1, \qquad g^{0r}(0, x) = 0, \qquad g^{rs}(0, x) = R^{-2}(t)\delta^{rs} \tag{2.12a}$$

$$u^\alpha(0, x) = \delta^\alpha{}_0 \tag{2.12b}$$

$$n(0, x) = n_0(t), \qquad T(0, x) = T_0(t) \tag{2.12c}$$

where δ^{rs}, $\delta^\alpha{}_0$, and similar symbols denote the Kronecker deltas.

Suppose that an exact solution $\mathcal{G}_0$ is known and suppose that we are interested in studying a situation for which the deviation of $\mathcal{G}_\epsilon$ from $\mathcal{G}_0$ is small. In perturbation theory (Ehlers, 1973; Wald, 1984), the parameter ϵ measures the size of perturbation in the sense that $\mathcal{G}_\epsilon(x)$ depends differentiably on ϵ for each $x \in X$ and $\mathcal{G}_0(x)$ is a background solution. Approximation methods aim at constructing $\mathcal{G}_\epsilon(x)$ for small ϵ, and such a construction is feasible if we can determine the time evolution of the following objects:

$$Q^{\alpha\beta} := \left(\frac{\partial g^{\alpha\beta}}{\partial \epsilon}\right)_{|\epsilon=0}, \qquad U^\alpha := \left(\frac{\partial u^\alpha}{\partial \epsilon}\right)_{|\epsilon=0} \tag{2.13a}$$

$$M := \frac{1}{n_0}\left(\frac{\partial n}{\partial \epsilon}\right)_{|\epsilon=0}, \qquad K := \frac{1}{T_0}\left(\frac{\partial T}{\partial \epsilon}\right)_{|\epsilon=0} \tag{2.13b}$$

It is convenient to think of

$$\delta\mathcal{G}_0 := (Q^{\alpha\beta}, U^\alpha, n_0 M, T_0 K) \tag{2.14}$$

as being the infinitesimal perturbation of $\mathcal{G}_0$.

Since $\{\mathcal{G}_\epsilon(x); \epsilon \in I\}$ is a one-parameter family of exact solutions of the full nonlinear field equations, we can derive a "closed" set of dynamical equations for $\delta\mathcal{G}_0$ directly from equations (2.1a) and (2.1b) by first differentiating them with respect to ϵ and then setting in the result ϵ equal to zero. However, before doing so, we first define the useful quantities D^{rs}, D, and F^{rs} by

$$D^{rs} := \tfrac{1}{2}R^2 Q^{rs}, \qquad D := \tfrac{1}{3}\delta_{rs}D^{rs} \tag{2.15a}$$

$$F^{rs} := D^{rs} - D\delta^{rs} \tag{2.15b}$$

Here, of course, F^{rs} is the second-rank, symmetric, traceless three-tensor. Differentiation of equations (2.1a) and (2.1b) with respect to ϵ at $\epsilon = 0$ then will yield the linear propagation equations we seek; they are of the form

$$\frac{2}{H}\dot{D} + \frac{2}{3H}Q^{0r}{}_{,r} + Q^{00} + \frac{1}{3R^2H^2}(F^{rs}{}_{,rs} - 2\delta^{rs}D_{,rs}) = -\frac{1}{3H^2}E \tag{2.16a}$$

$$(D)^{\cdot\cdot} + 3H\dot{D} + \frac{1}{2}H\dot{Q}^{00} + \left(\dot{H} + \frac{3}{2}H^2\right)Q^{00} + \frac{1}{6R^2}\delta^{rs}Q^{00}{}_{,rs}$$
$$+ \frac{1}{3}\dot{Q}^{0r}{}_{,r} + HQ^{0r}{}_{,r} + \frac{1}{6R^2}(F^{rs}{}_{,rs} - 2\delta^{rs}D_{,rs}) = \frac{1}{2}P \tag{2.16b}$$

$$\dot{F}^{rs}{}_{,s} - 2\delta^{rs}\dot{D}_{,s} - \frac{1}{2}(\delta^{rp}Q^{0s}{}_{,ps} - \delta^{ps}Q^{0r}{}_{,ps}) - H\delta^{rs}Q^{00}{}_{,s} = -2R^2\dot{H}(U^r + Q^{0r}) \tag{2.16c}$$

$$(F^{rs})^{\cdot\cdot} + 3H\dot{F}^{rs} + \frac{1}{2}(\delta^{rp}\dot{Q}^{0s}{}_{,p} + \delta^{sp}\dot{Q}^{0r}{}_{,p}) - \frac{1}{3}\delta^{rs}\dot{Q}^{0p}{}_{,p}$$

$$+ \frac{3}{2}H(\delta^{rp}Q^{0s}{}_{,p} + \delta^{sp}Q^{0r}{}_{,p}) - H\delta^{rs}Q^{0p}{}_{,p}$$

$$+ \frac{1}{2R^2}\left(\delta^{rp}\delta^{sq} - \frac{1}{3}\delta^{rs}\delta^{pq}\right)Q^{00}{}_{,pq}$$

$$= R^{-2}\left(\delta^{pq}F^{rs}{}_{,pq} - \delta^{rq}F^{sp}{}_{,pq} - \delta^{sq}F^{rp}{}_{,pq} + \frac{2}{3}\delta^{rs}F^{pq}{}_{,pq}\right)$$

$$+ R^{-2}\left(\delta^{rp}\delta^{sq} - \frac{1}{3}\delta^{rs}\delta^{pq}\right)D_{,pq} \tag{2.16d}$$

$$-3\dot{D} + \dot{M} + U^r{}_{,r} = 0 \tag{2.16e}$$

where a comma denotes the derivative of a "three-tensor" with respect to x^r, $r = 1, 2, 3$, and where E and P are given by

$$E := n_0 e_M M + T_0 e_T K, \qquad P := n_0 p_M M + T_0 p_T K \tag{2.17}$$

with

$$p_M := \partial p_0/\partial n_0, \qquad p_T := \partial p_0/\partial T_0 \tag{2.18}$$

As regards the definitions of e_M and e_T, see equations (2.9).

Here is the best place to mention that the above results are straightforward but rather tedious consequences of equations (2.9) introduced in Banach and Piekarski (1996a). For lack of space, we will not comment on the technical details leading to the derivation of equations (2.16); however, these details are available on request. At this point we only mention the following: in deriving the linearized field equations, use was made of the fact that $2U^0$ equals $-Q^{00}$ [see Banach and Piekarski (1996a), equations (4.10a) and (4.11)]. Moreover, we adopt the summation convention: if a Latin index appears twice in the same term, once as a subscript and once as a superscript, the sign Σ will be omitted (we recall that Latin indices range from 1 to 3, Greek indices from 0 to 3).

Now, we can verify that every solution of equations (2.16) obeys

$$\dot{E} - 2\dot{H}(U^r{}_{,r} - 3\dot{D}) + 3H(E + P) = 0 \tag{2.19}$$

A further study of equations (2.16) yields the supplementary balance law, interpreted as the equation of balance of U^r,

$$\dot{U}^r + \dot{Q}^{0r} + \left[5H + \frac{1}{\dot{H}}(H)^{\cdot\cdot}\right](U^r + Q^{0r}) + \frac{1}{2R^2}\,\delta^{rs}\left(Q^{00}{}_{,s} - \frac{1}{\dot{H}}P_{,s}\right) = 0 \tag{2.20}$$

These mathematical consequences of equations (2.16) can also be derived by directly differentiating the equation of motion of the matter $T^{\alpha\beta}{}_{;\beta} = 0$ with respect to ϵ at $\epsilon = 0$. In other words, equations (2.20) are obtainable from the balance law, which is a local conservation of energy and momentum. The background energy and pressure (e_0 and p_0) or their time derivatives ($\dot{e}_0$ and $\dot{p}_0$) do not appear in equations (2.20), because the unperturbed equations (2.5) tell us that these background quantities can be expressed in terms of H, $\dot{H}$, and $(H)^{\cdot\cdot}$.

Let $\delta\mathscr{G}_0$ or P denote the solution of equations (2.16). The analysis indicates that this solution is unique only up to a Lie derivative of the background solution with respect to an arbitrary vector field on the spacetime manifold. In a sense, the failure of equations (2.16) to produce the unique solution to the "Cauchy problem" is of no physical importance beyond the above comment, because we can easily obtain the equivalence class $[\delta\mathscr{G}_0]$ of $\delta\mathscr{G}_0$ once one choice of $\delta\mathscr{G}_0$ has been made. Moreover, equations (2.16) play a fundamental role in perturbation theory. One obvious reason for this is as follows: *there is an infinite number of exact differential consequences of equations (2.16) and some of these consequences describe the evolution of gauge-invariant variables. Clearly, the inverse statement is not true.*

On the other hand, the usual approach to the derivation of the equations governing linearized perturbations in cosmology does not aim at describing the equivalence class $[\delta\mathscr{G}_0]$ in a unique way. Rather, the basic purpose of this approach is to impose at the beginning a *synchronous gauge*,

$$Q^{00} = 0, \qquad Q^{0r} = 0 \qquad (\Rightarrow U^0 = 0) \tag{2.21}$$

or any other gauge condition to simplify the form of the metric and/or matter perturbations and then work only with the specific metric components and matter variables. However, as already observed by Bardeen (1980) and Ellis and Bruni (1989), this "naive" approach obscures the real situation. Indeed, "if the gauge condition imposed to simplify the metric leaves a residual gauge freedom, the perturbation equations will have spurious gauge mode solutions which can be completely annulled by a gauge transformation and have no physical reality" (Bardeen, 1980, p. 1882); "while if it is fully specified, its relation to what we really want to know (the spatial variation of density in the Universe) is convoluted and difficult to interpret" (Ellis and Bruni, 1989, p. 1804).

The only way out of this impasse is to drop any assumptions about the gauge and to look for a complete set of basic gauge-invariant quantities that code the information we need to describe the equivalence class $[\delta\mathcal{G}_0]$ in a unique way and to characterize density inhomogeneities in an almost-Robertson–Walker universe model.

2.3. Comments Concerning the Existence of Perturbation Theory

Cases are known in which linearized theory gives an incorrect description of the collection of solutions of the field equations near a fixed background solution. One such nontrivial example was given by Fischer *et al.* (1980), who considered the issue of linearization stability for the case of pure gravity, i.e., the existence of a one-parameter family of exact solutions corresponding to a solution of the linearized field equations. Among other things, they were able to prove that if the background space-time $(X, g^{\alpha\beta}_{(0)})$ possesses a compact, spacelike Cauchy hypersurface, then the vacuum Einstein field equations are linearization stable about $(X, g^{\alpha\beta}_{(0)})$ if and only if $(X, g^{\alpha\beta}_{(0)})$ does not possess a Killing vector field. If $(X, g^{\alpha\beta}_{(0)})$ has a Killing vector field, then it is necessary that the infinitesimal perturbations $Q^{\alpha\beta}$ of $g^{\alpha\beta}_{(0)}$ satisfy a quadratic integral constraint involving the Killing vector field in order that a one-parameter family $\{g^{\alpha\beta}(\epsilon, \bullet);\ \epsilon \in I\}$ corresponding to $Q^{\alpha\beta}$ does exist. This imposes further conditions containing only the first-order variations $Q^{\alpha\beta}$. Thus in some sense the linearized theory of pure gravity near a space-time with symmetry is not sufficient to capture the dominant effects of the nonlinear theory. However, as remarked already by Wald (1984, p. 187), "linearization stability is believed to hold for asymptotically flat perturbations of all asymptotically flat background space-times (even if Killing fields are present), although this has been proven only for the flat background space-time."

In the context of cosmology, D'Eath (1976) has examined the nature of full nonlinear perturbations of the Robertson–Walker universes, together with their relation to solutions of the linearized field equations. In this case the unperturbed and perturbed solutions contain matter. Thus there is no instability in the sense of Fischer *et al.* (1980) or Brill and Deser (1973), for essentially obvious reasons. Moreover, physically reasonable solutions of the linearized field equations do indeed correspond to solutions of the full nonlinear equations near the Robertson–Walker backgrounds. Only the collection of solutions of the field equations near the $k = +1$ background has unusual features, and the linearized treatment (while certainly useful) does not allow one to understand these features completely. In the $k = -1$ case, matter quantities may vary freely within uniform bounds over the initial surface. If $k = 0$, one has to consider perturbations which die away at large distances. Further interesting details can be found in D'Eath (1976).

3. THE GAUGE PROBLEM AND ITS SOLUTION

3.1. A Complete Set of Basic Gauge-Invariant Variables

Because of a gauge freedom in general relativity corresponding to the group of diffeomorphisms of space-time (Hawking and Ellis, 1973), two infinitesimal perturbations $\delta\mathcal{G}_0$ and $\delta\mathcal{G}_0'$ represent the same perturbation if (and only if) there is a vector field v on the space-time manifold X such that

$$\delta\mathcal{G}_0' = \delta\mathcal{G}_0 + \mathcal{L}_v\mathcal{G}_0 \tag{3.1}$$

where $\mathcal{L}_v\mathcal{G}_0$ denotes the Lie derivative of a background solution $\mathcal{G}_0$ with respect to v. Strictly speaking, then, the object of most physical interest is not just one perturbation $\delta\mathcal{G}_0$, but a whole equivalence class $[\delta\mathcal{G}_0]$ of all perturbations $\delta\mathcal{G}_0'$ which are equivalent to $\delta\mathcal{G}_0$. These definitions do not tell us directly how to use $[\delta\mathcal{G}_0]$ in practical calculations, or whether such calculations are possible at all. However, there is a definite description of $[\delta\mathcal{G}_0]$ in terms of a complete set

$$\mathrm{D} := \{\chi, \Gamma, \Omega, \Omega^r, \Delta, \Delta^{rs}, S^{ijrs}\} \tag{3.2}$$

of basic gauge-invariant variables. This set, which we derived in Section 5.2 of Banach and Piekarski (1996a), may therefore be thought of as representing the equivalence class $[\delta\mathcal{G}_0]$ of $\delta\mathcal{G}_0$:

$$[\delta\mathcal{G}_0] \Leftrightarrow \mathrm{D} \tag{3.3}$$

The correspondence (3.3) defines a "coordinate system" on the quotient space $\mathcal{P}/\mathcal{P}_0$ to which $[\delta\mathcal{G}_0]$ belongs. [See Banach and Piekarski (1996a) for the precise definition of this quotient space.] Thus one can construct a mapping A which is a bijection from $\mathcal{P}/\mathcal{P}_0$ onto the "vector space" $\mathcal{D}$ such that if $[\delta\mathcal{G}_0] \in \mathcal{P}/\mathcal{P}_0$, then $\mathrm{D} = A([\delta\mathcal{G}_0])$ is a set of basic gauge-invariant quantities associated with $[\delta\mathcal{G}_0]$. In Banach and Piekarski (1996a, Section 6) we often denote the gauge-invariant perturbation $[\delta\mathcal{G}_0]$ by P; there, we also name $\mathcal{D}$ the image space of $\mathcal{P}/\mathcal{P}_0$. It should be obvious now that one can extract $[\delta\mathcal{G}_0] \in \mathcal{P}/\mathcal{P}_0$ from $\mathrm{D} \in \mathcal{D}$ in a unique way and conversely. Another welcome feature is that any complicated gauge-invariant quantity can be constructed directly from D through purely algebraic and differential operations (Banach and Piekarski, 1996a).

The explicit expressions for the gauge-invariant variables in D have a simple form if given in terms of rescaled perturbations $\{Q^{00}, Q^{0r}, D, F^{rs}, U^\alpha, M, K\}$ defined by equations (2.13) and (2.15); these expressions take the form (Banach and Piekarski, 1996a)

$$\chi := Q^{00} + 2U^0 = 0 \tag{3.4a}$$

$$\Gamma := K + (3HT_0)^{-1}\dot{T}_0 M \tag{3.4b}$$

$$\Omega := -\tfrac{1}{2}Q^{00} + \tfrac{1}{3}H^{-2}(\dot{H}M - H\dot{M}) \quad (3.4c)$$

$$\Omega^r := -3R^2HQ^{0r} - 3R^2HU^r + \delta^{rs}\frac{\partial M}{\partial x^s} \quad (3.4d)$$

$$\Delta := -\frac{3}{2}Q^{00} - \frac{3}{H}\dot{D} + \frac{1}{H}U^r{}_{,r} + H^{-2}\dot{H}M \quad (3.4e)$$

$$\Delta^{rs} := \frac{1}{H}\dot{F}^{rs} - \frac{1}{2H}(\delta^{rp}U^s{}_{,p} + \delta^{sp}U^r{}_{,p}) + \frac{1}{3H}U^p{}_{,p}\delta^{rs} \quad (3.4f)$$

$$S^{ijrs} := \delta^{sq}\delta^{p[i}Z^{j]r}{}_{,pq} - \delta^{rq}\delta^{p[i}Z^{j]s}{}_{,pq} \quad (3.4g)$$

where

$$Z^{rs} := 2(\tfrac{1}{3}M - D)\delta^{rs} - 2F^{rs} \quad (3.5)$$

The importance of D is illustrated by the following property (Banach and Piekarski, 1996a): two infinitesimal perturbations $\delta\mathcal{G}_0$ and $\delta\mathcal{G}_0'$ are equivalent if (and only if) they determine one common set of basic gauge-invariant variables, i.e., if (and only if) D equals D′. Our conclusion, then, is that $[\delta\mathcal{G}_0]$ can be identified with D. Also, as we have already remarked, with equations (3.4) we have a set of basic variables which enables us to calculate other gauge-invariant quantities directly from D through purely algebraic and differential operations. In this sense, then, the set D is complete. Further details concerning these problems are given in the Appendix of Banach and Piekarski (1996a); see also Section 5 of this paper.

We are now in a position to derive useful propagation equations for the basic gauge-invariant variables.

3.2. Propagation Equations in Gauge-Invariant Variables

A set of deterministic equations can be obtained directly in terms of the basic gauge-invariant variables defined in Section 3.1. These equations are attractive for at least four reasons: (a) they lead to a unique solution of the "Cauchy problem;" (b) their form is independent of the gauge chosen; (c) any gauge-invariant perturbation $D \in \mathfrak{D}$ is a solution of these equations; and (d) none of the solutions of equations (3.6) can be annulled by a gauge transformation. Nevertheless, it is perhaps important to stress that the gauge invariance in itself does not completely resolve the ambiguity of what one means by a physical solution to the linearized field equations, and some additional arguments are always necessary in order to eliminate those mathematical solutions of gauge-invariant equations which do not belong to the image space $\mathfrak{D}$ of $\mathcal{P}/\mathcal{P}_0$. These arguments are given in Section 4.2. We also prove there that such problems cannot be avoided and thus are explicitly or implicitly present in any gauge-invariant approach to cosmological perturbations.

One can obtain exact propagation equations for the basic variables D, defined by equation (3.2), from equations (2.5), (2.8), (2.16), and (3.4); as a result we have

$$\dot{\Gamma} = \frac{3}{e_T}\left[\frac{1}{T_0}(e_0 - n_0 e_M + p_0)\left(1 + T_0\frac{e_{TT}}{e_T}\right) - (e_T + p_T - n_0 e_{MT})\right]H\Gamma \tag{3.6a}$$

$$\dot{\Omega} + 2\left(H + \frac{1}{H}\dot{H}\right)\Omega - \frac{1}{3R^2 H}\left[1 + \frac{1}{3H\dot{H}}(H)^{\cdot\cdot}\right]\Omega^r{}_{,r}$$

$$= -\frac{1}{6H}T_0(e_T + 3p_T)\Gamma + \frac{T_0 p_T}{6R^2 H\dot{H}}\delta^{rs}\Gamma_{,rs} \tag{3.6b}$$

$$(\Omega^r)^{\cdot\cdot} + \left[5H + \frac{1}{\dot{H}}(H)^{\cdot\cdot}\right]\dot{\Omega}^r + \left[6H^2 + 4\dot{H} + 2\frac{1}{\dot{H}}H(H)^{\cdot\cdot}\right.$$

$$\left. - \frac{1}{\dot{H}^2}[(H)^{\cdot\cdot}]^2 + \frac{1}{\dot{H}}(H)^{\cdot\cdot\cdot}\right]\Omega^r + \frac{1}{R^2}\left[1 + \frac{1}{3H\dot{H}}(H)^{\cdot\cdot}\right]\delta^{rs}\Omega^p{}_{,ps}$$

$$= -\frac{T_0 p_T}{2R^2\dot{H}}\delta^{rs}\delta^{pq}\Gamma_{,pqs} + \frac{3}{2}T_0\left\{\frac{1}{3}e_T - \left[1 - \frac{1}{\dot{H}}H^2 - \frac{1}{\dot{H}^2}H(H)^{\cdot\cdot}\right]p_T\right.$$

$$+ \frac{3H^2 p_T}{\dot{H}e_T}\left[p_T - n_0 e_{MT} - (e_0 - n_0 e_M + p_0)\frac{e_{TT}}{e_T}\right]$$

$$\left. + \frac{3H^2}{\dot{H}}\left[n_0 p_{MT} + (e_0 - n_0 e_M + p_0)\frac{p_{TT}}{e_T}\right]\right\}\delta^{rs}\Gamma_{,s} \tag{3.6c}$$

$$\Delta = 3\Omega \tag{3.6d}$$

$$(\Delta^{rs})^{\cdot\cdot} + \left(5H + 2\frac{\dot{H}}{H}\right)\dot{\Delta}^{rs} + \left[6H^2 + 8\dot{H} + \frac{1}{H}(H)^{\cdot\cdot}\right]\Delta^{rs}$$

$$- \frac{1}{2H}(\delta^{rp}\dot{W}^s{}_{,p} + \delta^{sp}\dot{W}^r{}_{,p}) + \frac{1}{3H}\delta^{rs}\dot{W}^p{}_{,p}$$

$$- \frac{3}{2}(\delta^{rp}W^s{}_{,p} + \delta^{sp}W^r{}_{,p}) + \delta^{rs}W^p{}_{,p}$$

$$= R^{-2}\left(\delta^{pq}\Delta^{rs}{}_{,pq} - \frac{3}{4}\delta^{rq}\Delta^{sp}{}_{,pq}\right.$$

$$- \frac{3}{4} \delta^{sp} \Delta^{rp}{}_{,pq} + \frac{1}{2} \delta^{rs} \Delta^{pq}{}_{,pq} \Big) - \frac{1}{24R^4H^2} \delta^{pq} (\delta^{rk} \Omega^s{}_{,kpq} + \delta^{sk} \Omega^r{}_{,kpq})$$

$$+ \frac{1}{12R^4H^2} \delta^{rk} \delta^{sn} \Omega^p{}_{,pkn}$$

$$\dot{S}^{ijrs} = -2H(\delta^{sq} \delta^{p[i} \Delta^{j]r}{}_{,pq} - \delta^{rq} \delta^{p[i} \Delta^{j]s}{}_{,pq}) \tag{3.6f}$$

where

$$e_{MT} := \frac{\partial^2 e_0}{\partial n_0 \partial T_0}, \qquad e_{TT} := \frac{\partial^2 e_0}{\partial T_0^2} \tag{3.7a}$$

$$p_{MT} := \frac{\partial^2 p_0}{\partial n_0 \partial T_0}, \qquad p_{TT} := \frac{\partial^2 p_0}{\partial T_0^2} \tag{3.7b}$$

$$W^r := -R^{-2} \left[1 + \frac{1}{3H\dot{H}} (H)^{\cdot\cdot} \right] \Omega^r - \frac{T_0 p_T}{2R^2 \dot{H}} \delta^{rs} \Gamma_{,s} \tag{3.8}$$

With equations (3.7) we find that the form of the coefficients e_{MT}, e_{TT}, and similar objects may be determined directly from the equations of state for the background perfect fluid.

The idea of using such gauge-invariant quantities and propagation equations was originally proposed by Gerlach and Sengupta (1978b), and initial aspects of our formalism (Banach and Piekarski, 1996a) have been developed in various papers [see, e.g., Ehlers (1973) and Wald (1984)]. As far as we are aware, the full set of equations given here has not been obtained before, although Ellis *et al.* (1989) go a long way toward it in the case of a barotropic perfect fluid where p and e are functionally dependent: $p = p(e)$; see their discussion directly after equation (21), p. 1821. Also, as demonstrated in Section 6, our equations are related to those obtained by the harmonic decomposition of three-fields [see, e.g., Bardeen (1980) and Gerlach and Sengupta (1978a)] but are, we believe, more fundamental and more convenient in many applications.

Gauge-invariant variables provide a focal point for discussing density inhomogeneities in an almost-Robertson–Walker universe model. In fact, of the equations presented, the one of most physical interest is that for Ω^r. As we shall soon see, the gauge-invariant quantity Ω^r and its magnitude most closely correspond to the intention of the usual M in representing the fractional density increase in a comoving density fluctuation. In fact, to first order in the deviations from the background solution, the gauge-invariant quantity $n_0 R^{-2} \Omega^r$ can be identified with the spatial gradient of the number density n. Thus equation (3.6c) is the basic result of this paper. It is a differential equation determining the evolution of Ω^r along the fluid flow lines, equivalent to the central results of Bardeen (1980) and Ellis *et al.* (1989); see equation (4.9) in Bardeen (1980) and equation (28) in Ellis *et al.* (1989).

An important feature of the above discussion is a demonstration that, for the case of a general perfect fluid with nonzero pressure, the evolution of the fields Ω^r and $\Gamma_{,r}$ satisfying equations (3.6c) and

$$\dot{\Gamma}_{,r} = \frac{3}{e_T}\left[\frac{1}{T_0}(e_0 - n_0 e_M + p_0)\left(1 + T_0 \frac{e_{TT}}{e_T}\right) - (e_T + p_T - n_0 e_{MT})\right] H\Gamma_{,r} \tag{3.9}$$

is exactly decoupled from the evolution of Ω, Δ, Δ^{rs}, and S^{ijrs}. Thus, if one finds how to define Ω^r and the gradient of Γ directly from the observations (see Section 5), the resulting simplification can be enormous. Finally, it is quite clear in our analysis that the evolution of Γ (and hence of $\Gamma_{,r}$) is completely independent of the wavelength of the fluctuation. This is in contrast with the case of equations (3.6b)–(3.6f), for which, as shown below, such effects cannot be ignored.

Among the problems that can be studied with this sort of approach, the examination of the effect of inhomogeneities on the time development of perturbations presents a most interesting challenge. Thus, for example, it would be important to provide a *rigorous* derivation of the Jeans instability via a linearized system of equations and to explicitly show that the evolution of Ω^r is affected by particle horizons. Clearly, we here concern ourselves with situations where the temperature does not vanish in the background ($T_0 \neq 0$); otherwise the Jeans criterion is irrelevant to this evolution, whether we consider large- or small-scale inhomogeneity, because the individual world lines evolve independently [see, e.g., Banach and Piekarski (1994c, Section 6) or Section 4 of this paper]. In any case, equations (3.6c) and (3.9) will find their most interesting applications in considerations of the gauge-invariant quantity Ω^r for large values of the wave vector, away from the usual hydrodynamic regime where more conventional methods are successful.

4. UNIVERSES WITH A PRESSURE-FREE BACKGROUND FLUID

4.1. Equations Describing the Perturbation of a Flat Robertson–Walker Model

Matter is treated here as an assemblage of material particles, all having the same proper mass *m*, which in the case of a chemically inert fluid might be a hydrogen gas during the dust-dominated epoch, from a redshift $Z \cong 1000$ until $Z \cong 30$ or so. Under these assumptions, we can compare the hydrodynamic model with a full kinetic-theory description (Ellis *et al.*, 1983; Banach and Piekarski, 1994a–c; Banach and Makaruk, 1995). But the latter provides a most fundamental route to calculating the energy density and pressure and to verifying that physically, and in some sense mathematically, natural equations of state are of the form (Banach and Piekarski, 1994a–c)

$$e = 4\pi^{-1/2} mn \int_{\mathrm{R}} z^2 \left(1 + 2 \frac{k_B T}{m} z^2\right)^{1/2} \exp(-z^2)\, dz \tag{4.1a}$$

$$p = \frac{8}{3} \pi^{-1/2} n k_B T \int_{\mathrm{R}} z^4 \left(1 + 2 \frac{k_B T}{m} z^2\right)^{-1/2} \exp(-z^2)\, dz \tag{4.1b}$$

where k_B is the Boltzmann constant.

Now implement a time-dependent canonical transformation from the original gauge-invariant variable Γ to a new gauge-invariant variable $\mathbb{K}$ obeying

$$\mathbb{K} := \frac{1}{M} k_B T_0 \Gamma \tag{4.2}$$

Turning our attention back to equations (2.8) and (3.6a), we then see that $\mathbb{K}$ will satisfy a transformed equation of the form

$$\dot{\mathbb{K}} = \frac{3}{e_T} \left[(e_0 - n_0 e_M + p_0) \frac{e_{TT}}{e_T} - (e_T + p_T - n_0 e_{MT}) \right] H \mathbb{K} \tag{4.3}$$

After a bit of mathematical manipulation which employs only the equations of state (4.1a) and (4.1b), we find from equations (2.8) and (4.3) that the following equations hold:

$$\dot{T}_0 = -2HT_0 \tag{4.4a}$$

$$\dot{\mathbb{K}} = -2H\mathbb{K} \tag{4.4b}$$

Thus, because of the assumptions (4.1a) and (4.1b), T_0 and $\mathbb{K}$ evolve at the same rate as R^{-2}, i.e., $T_0 R^2$ = const and the value of $\mathbb{K} R^2$ is independent of time.

In a similar fashion, using equations (4.1) and (4.2), we can transform the remaining propagation equations in the system (3.6). These transformed equations have the advantage (as compared with the original ones) that, for $T_0 = 0$, they take a very simple form. More specifically, if the temperature vanishes in the background, equations (3.6) simplify to

$$\dot{\mathrm{K}} + \frac{4}{3t}\,\mathrm{K} = 0 \tag{4.5a}$$

$$\dot{\Omega} - \frac{2}{3t}\,\Omega = -\frac{3}{2t}\,\mathrm{K} - \frac{t}{2R^2}\,\delta^{rs}\mathrm{K}_{,rs} \tag{4.5b}$$

$$(\Omega^r)^{\cdot\cdot} + \frac{4}{3t}\,\dot{\Omega}^r - \frac{2}{3t^2}\,\Omega^r = \frac{1}{R^2}\,\delta^{rs}\delta^{pq}\mathrm{K}_{,pqs} - \frac{1}{t^2}\,\delta^{rs}\mathrm{K}_{,s} \tag{4.5c}$$

$$\Delta = 3\Omega \tag{4.5d}$$

$$\begin{aligned}(\Delta^{rs})^{\cdot\cdot} &+ \frac{4}{3t}\,\dot{\Delta}^{rs} - \frac{2}{3t^2}\,\Delta^{rs} + \frac{1}{R^2}\,\delta^{rp}\delta^{sq}\mathrm{K}_{,pq} - \frac{1}{3R^2}\,\delta^{rs}\delta^{pq}\mathrm{K}_{,pq} \\ &= R^{-2}\left(\delta^{pq}\Delta^{rs}{}_{,pq} - \frac{3}{4}\,\delta^{rq}\Delta^{sp}{}_{,pq} - \frac{3}{4}\,\delta^{sq}\Delta^{rp}{}_{,pq} + \frac{1}{2}\,\delta^{rs}\Delta^{pq}{}_{,pq}\right) \\ &\quad - \frac{3t^2}{32R^4}\,\delta^{pq}(\delta^{rk}\Omega^{s}{}_{,kpq} + \delta^{sk}\Omega^{r}{}_{,kpq}) + \frac{3t^2}{16R^4}\,\delta^{rk}\delta^{sn}\Omega^{p}{}_{,pkn}\end{aligned} \tag{4.5e}$$

$$\dot{S}^{ijrs} = -\frac{4}{3t}\,(\delta^{sq}\delta^{p[i}\Delta^{j]r}{}_{,pq} - \delta^{rq}\delta^{p[i}\Delta^{j]s}{}_{,pq}) \tag{4.5f}$$

where the expansion factor R is given by

$$R(t) = \mathrm{C}t^{2/3} \tag{4.6a}$$

with

$$\mathrm{C} := (3\pi m\vartheta)^{1/3}, \qquad \vartheta = \mathrm{const} \tag{4.6b}$$

Clearly, one also deduces from equations (2.5a) and (4.1a) that the background number density n_0 is related to the cosmic time t by

$$n_0(t) = \frac{4}{3mt^2} \tag{4.7}$$

The analysis simplifies still further when the fluid, already assumed to be pressure free in the background ($T_0 = 0$ implies $p_0 = 0$), is also pressure free in the perturbed space-time ($\mathrm{K} = 0$). Equation (4.5c) then becomes the standard equation for zero-pressure density perturbation growth relative to proper time along the flow lines in an expanding universe, obtained by Lifshitz (1946) in his pioneering study of the instability of Robertson–Walker universe models.

One final word regarding the physical interpretation of K. If $T_0 = 0$, we immediately find from equations (2.13b), (3.4b), and (4.2) that

$$\mathrm{K} = \frac{1}{m} k_{\mathrm{B}} \left(\frac{\partial T}{\partial \epsilon} \right)_{|\epsilon=0} \tag{4.8}$$

Thus K measures the relative size of the temperature perturbation δT relative to the Boltzmann temperature m/k_{B}.

4.2. Solutions of the Perturbed Equations

Here it will be convenient to restrict attention to the so-called scalar perturbations. We assume that this notion is well understood in perturbation theory. Nevertheless, we refer the interested reader to Mukhanov *et al.* (1992, p. 212) for a detailed definition of scalar, vector, and tensor perturbations. The (scalar) solution of equations (4.5) is given by

$$\mathrm{K} = d_1 t^{-4/3} \tag{4.9a}$$

$$\Omega = \frac{3}{4} d_1 t^{-4/3} + \frac{3}{8\mathrm{C}^2} \delta^{pq} (d_1 t^{-2/3})_{,pq} + d_2 t^{2/3} \tag{4.9b}$$

$$\Omega^r = -\frac{3}{2} \delta^{rs} \left[d_1 t^{-4/3} + \frac{3}{2\mathrm{C}^2} \delta^{pq} (d_1 t^{-2/3})_{,pq} \right]_{,s} + \delta^{rs} (d_3 t^{-1} + d_4 t^{2/3})_{,s} \tag{4.9c}$$

$$\Delta = \frac{9}{4} d_1 t^{-4/3} + \frac{9}{8\mathrm{C}^2} \delta^{pq} (d_1 t^{-2/3})_{,pq} + 3 d_2 t^{2/3} \tag{4.9d}$$

$$\begin{aligned} \Delta^{rs} = {} & \delta^{rp} \delta^{sq} \left(d_5 t^{-1} + \frac{9}{4\mathrm{C}^2} d_1 t^{-2/3} + d_6 t^{2/3} \right)_{,pq} \\ & - \frac{1}{3} \delta^{rs} \delta^{pq} \left(d_5 t^{-1} + \frac{9}{4\mathrm{C}^2} d_1 t^{-2/3} + d_6 t^{2/3} \right)_{,pq} \end{aligned} \tag{4.9e}$$

$$\begin{aligned} S^{ijrs} = {} & [(\delta^{sq} \delta^{p[i} \delta^{j]r} - \delta^{rq} \delta^{p[i} \delta^{j]s}) d_7]_{,pq} \\ & - \frac{2}{3} \delta^{kn} \left[\delta^{sq} \delta^{p[i} \delta^{j]r} \left(\frac{2}{3} d_5 t^{-1} + \frac{9}{4\mathrm{C}^2} d_1 t^{-2/3} - d_6 t^{2/3} \right) \right. \\ & \left. - \delta^{rq} \delta^{p[i} \delta^{j]s} \left(\frac{2}{3} d_5 t^{-1} + \frac{9}{4\mathrm{C}^2} d_1 t^{-2/3} - d_6 t^{2/3} \right) \right]_{,pqkn} \end{aligned} \tag{4.9f}$$

where C is a constant satisfying equation (4.6b). Included in this solution are seven arbitrary functions of spatial positions (denoted d_1, d_2, . . . , and d_7), such that for $d_1 = 0$ and $d_3 = 0$ the gauge-invariant quantity Ω^r is proportional to $t^{2/3}$. If $d_1 = 0$, equation (4.9c) becomes the standard equation for zero-pressure or zero-temperature density perturbation growth relative to proper time along the flow lines in an expanding universe, giving the expected modes with powers of -1 and 2/3. From $d_3 = 0$, $d_4 = 0$, and equation (4.9c) it follows that Ω^r goes as $t^{-2/3}$, showing that there is also an extra decaying mode in Ω^r in the physical case, that is, if $d_1 > 0$. Moreover, at the late stages of cosmic expansion where it is physically reasonable to assume that $p_0 = 0$, there are no solutions of the "bounded and oscillatory" type. Thus in the case considered here (vanishing background pressure), the evolution of $\mathrm{K}, \Omega, \Omega^r, \Delta, \Delta^{rs}$, and S^{ijrs} is independent of the wavelength of the perturbation.

One knows full well that linear propagation equations for the gauge-invariant variables are always obtained by performing some complicated algebraic and differential operations on equations (2.16). Thus the crucial remaining task in interpreting the results of any gauge-invariant theory based on the assumption $p_0 = 0$ or $T_0 = 0$ is to verify whether all solutions D of equations (4.5) are elements of the image $\mathfrak{D}$ of $\mathcal{P}/\mathcal{P}_0$ under A, i.e., whether D equals $A([\delta\mathcal{G}_0])$ for some $[\delta\mathcal{G}_0]$. Here we recall that A defines a "coordinate system" on the quotient space $\mathcal{P}/\mathcal{P}_0$ and that $\mathfrak{D} := A(\mathcal{P}/\mathcal{P}_0)$. In fact, part of the problem involves determining the fields $\mathrm{K}, \Omega, \Omega^r, \Delta, \Delta^{rs}, S^{ijrs}$ consistent with the definitions (3.4), (4.2), and the condition that the original quantities Q^{00}, Q^{0r}, D, F^{rs}, U^r, M, and K must satisfy equations (2.16) specialized to the $T_0 = 0$ case; such complexity is the price one pays for eliminating the gauge-dependent variables as unknowns.

Consequently, we now follow an alternative procedure for deriving the time development of gauge-invariant variables $\mathrm{K}, \Omega, \dots, S^{ijrs}$. The procedure is simply this. First, we specialize equations (2.16) to the case of vanishing background temperature. The corresponding equations and solutions may be found, e.g., in Banach and Piekarski (1994c), pp. 5897 and 5902, equations (5.5) and (6.1). Here, in order to obtain a particularly simple member of the equivalence class $[\delta\mathcal{G}_0]$ of $\delta\mathcal{G}_0$, it will be convenient to work in synchronous gauge ($Q^{00} = Q^{0r} = U^0 = 0$). Clearly, in this program, after solving equations (2.16) for the $T_0 = 0$ case, the evolution of basic gauge-invariant variables can be found directly from the definitions (3.4). *It is easily shown that this evolution is completely independent of the gauge chosen.*[3] The resulting formulas for $\mathrm{K}, \Omega, \dots, S^{ijrs}$ are then compared with the results (4.9). In view of what we have explained above, it is possible to prove that the gauge-invariant equations (4.9) give physically well-defined modes of growth and decay if and only if the following additional conditions hold:

[3] Suppose that the symbols D, E_{rs}, U^r, N, and R_0 have the same meaning as in Banach and Piekarski (1994c). Then the following relations are satisfied: $D = -\mathrm{D}$, $F^{rs} = -\mathrm{E}_{rs}$, $U^r = \mathrm{U}^r$, $M = \mathrm{N}$, and $R = R_0$. Comparison of equation (4.8) with equation (4.6c) of Banach and Piekarski (1994c) shows that K denotes the relative size of the temperature perturbation δT relative to the Boltzmann temperature m/k_B ($c = 1$). Here and in Banach and Piekarski (1994c) the constant C is given by equation (4.6b).

$$d_1 = \frac{1}{9} c_3, \qquad d_2 = -\frac{1}{8C^2} \delta^{rs} \frac{\partial^2 c_1}{\partial x^r \partial x^s} \tag{4.10a}$$

$$d_3 = -\frac{2}{3} \delta^{rs} \frac{\partial^a d_5}{\partial x^r \partial x^s}, \qquad d_4 = \frac{3}{20C^2} \delta^{rs} \frac{\partial^2 c_1}{\partial x^r \partial x^s} \tag{4.10b}$$

$$d_6 = \frac{3}{20C^2} c_1, \qquad d_7 = -\frac{1}{3} c_1 \tag{4.10c}$$

The coefficients c_1 and c_3 are two functions of spatial positions; these coefficients have exactly the same meaning as those appearing in equations (6.1) of Banach and Piekarski (1994c). Of course, one wishes to have some idea of the allowed spatial dependence of c_1 and c_3. However, this can only be found by considering the full nonlinear field equations. The form of $d_1, \ldots,$ d_7 follows directly from the form of $c_1 c_3$, and d_5 as a consequence of the identities (4.10); thus only three of the seven coefficients are independent. This also suggests that in each particular case one should carefully check on the properties of equations (2.16) and (3.6), because it is quite probable that some of the solutions to equations (3.6) are not consistent with the definitions of basic gauge-invariant variables.

These problems are also present in other gauge-invariant approaches to cosmological perturbations. Indeed, in a companion paper (Banach and Piekarski, 1996a) we demonstrated that if $\mathcal{K}$ is an arbitrary gauge-invariant quantity, then this quantity can be constructed directly from the basic variables through purely algebraic and differential operations. Thus the time development of $\mathcal{K}$ is always obtainable from the generic system of gauge-invariant equations. For nonbarotropic perfect fluids, this system consists of equations (3.6). Therefore, it should not be surprising that the general method for deducing the existence of physical and unphysical solutions for $\mathcal{K}$ must coincide with that already made familiar. This method is successful because it enables us to identify those solutions of the generic system which are elements of the image space $\mathcal{D}$ of $\mathcal{P}/\mathcal{P}_0$; *such solutions always exist.*

As explained at the beginning of Section 4.2, this discussion refers only to the scalar part of D. However, if the vector and tensor solutions of equations (3.6) and (4.5) are taken into account (Mukhanov *et al.*, 1992, p. 212), then our conclusions remain basically unchanged. Examples illustrating the above can also be given. (For lack of space, we will not consider them in this article.) Another remark is also in order. A special case of equations (4.9) is obtained by setting c_1 = const, c_3 = const, and d_5 const. Because of these assumptions, we easily find that the perturbed metric is also of Robertson–Walker form and that there are no growing hydrodynamic modes. Obviously, we must postulate that $c_3 > 0$; in other words, we must seek solutions for which the pressure or the temperature is nonzero. If $c_3 = 0$, a homogeneous scalar perturbation is really no perturbation at all, but an inappropriate choice of the background. If $c_3 > 0$, appeal to the theory of perturbations at the level c_1 = const, c_3 = const, and d_5 = const is both legitimate and useful.

5. THE PHYSICAL MEANING OF BASIC GAUGE-INVARIANT QUANTITIES

5.1. The Key Variables

Let $\{A(x, \epsilon); \epsilon \in I\}$ be a curve of geometrical objects (matter variables, tensor fields, etc.), and suppose that $A(x, \epsilon)$ depends differentiably on ϵ. As explained already by Ehlers (1973), the infinitesimal perturbation of $A_0 := (A)_{|\epsilon=0}$ is invariant under gauge transformations if and only if for all generating vector fields v, the Lie derivative of A_0 with respect to v equals zero:

$$\mathcal{L}_v A_0 = 0 \tag{5.1}$$

What are natural geometrical objects $A(\epsilon, x)$ satisfying equation (5.1) for all v? In discussing perturbations away from a $k = 0$ Robertson–Walker universe model, the only case of physical interest is a scalar A that is constant in the unperturbed space-time $(X, g_{(0)})$, or any tensor A that vanishes in $(X, g_{(0)})$. Also, a constant linear combination of products of Kronecker deltas $\delta^\alpha{}_\beta$ is acceptable from this point of view (Stewart and Walker, 1974); here, however, no such combination occurs naturally.

We can use the definition of a projection tensor into the tangent three-spaces orthogonal to u^α,

$$h^{\alpha\beta} := g^{\alpha\beta} + u^\alpha u^\beta \tag{5.2}$$

to write down a list of all the simple tensor fields A satisfying $A_0 = 0$; these tensor fields are described as follows.

1. The orthogonal spatial gradients of n, T, and the expansion $\Theta := u^\alpha{}_{;\alpha}$:

$$X^\alpha := h^{\alpha\beta} n_{,\beta}, \qquad Y^\alpha := h^{\alpha\beta} T_{,\beta} \tag{5.3a}$$

$$Z^\alpha := h^{\alpha\beta} \Theta_{,\beta} \tag{5.3b}$$

2. The vorticity, shear, and acceleration:

$$\omega_{\alpha\beta} := h_\alpha{}^\mu h_\beta{}^\nu u_{[\mu;\nu]} \tag{5.4a}$$

$$\sigma_{\alpha\beta} := h_\alpha{}^\mu h_\beta{}^\nu u_{(\mu;\nu)} - \tfrac{1}{3}\Theta h_{\alpha\beta} \tag{5.4b}$$

$$\dot{u}^\alpha := u^\alpha{}_{;\beta} u^\beta \tag{5.4c}$$

3. The electric and magnetic parts $E_{\alpha\beta}$, $H_{\alpha\beta}$ of the Weyl tensor $C_{\alpha\mu\beta\nu}$:

$$E_{\alpha\beta} := C_{\alpha\mu\beta\nu}u^{\mu}u^{\nu}, \qquad H_{\alpha\beta} := \tfrac{1}{2}C_{\alpha\mu\gamma\nu}u^{\mu}\eta^{\gamma\nu}{}_{\beta\sigma}u^{\sigma} \tag{5.5}$$

These are the simplest tensor fields which vanish in a $k = 0$ Robertson–Walker universe model. Thus we can easily find some "simple" gauge-invariant quantities by first differentiating these tensor fields with respect to ϵ and then setting ϵ equal to zero; in particular, denoting by δA the derivative of A with respect to ϵ at $\epsilon = 0$, we obtain

$$\delta X^0 = 0, \qquad \delta X^r = n_0 R^{-2}\Omega^r \tag{5.6}$$

$$\delta Y^0 = 0, \qquad \delta Y^r = T_0 R^{-2}\delta^{rs}\Gamma_{,s} - \frac{1}{3R^2H}\dot{T}_0\Omega^r \tag{5.7}$$

$$\delta Z^0 = 0, \qquad \delta Z^r = R^{-2}H\delta^{rs}\Delta_{,s} - \frac{1}{R^2H}\dot{H}\Omega^r \tag{5.8}$$

$$\delta\omega_{00} = 0, \qquad \delta\omega_{0r} = 0 \tag{5.9a}$$

$$\delta\omega_{rs} = \frac{1}{6H}(\delta_r{}^q\delta_{sp} - \delta_{rp}\delta_s{}^q)\Omega^p{}_{,q} \tag{5.9b}$$

$$\delta\sigma_{00} = 0, \qquad \delta\sigma_{0r} = 0 \tag{5.10a}$$

$$\delta\sigma_{rs} = -R^2H\delta_{rp}\delta_{sq}\Delta^{pq} \tag{5.10b}$$

$$\delta\dot{u}^0 = 0, \qquad \delta\dot{u}^r = -R^{-2}\delta^{rs}\Omega_{,s} - \frac{1}{3R^2H}\dot{\Omega}^r + \frac{1}{3R^2H^2}\dot{H}\Omega^r \tag{5.11}$$

$$\delta E_{00} = 0, \qquad \delta E_{0r} = 0 \tag{5.12a}$$

$$\begin{aligned}\delta E_{rs} = {} & \frac{1}{2}R^2H\delta_{rp}\delta_{sq}\dot{\Delta}^{pq} + \frac{1}{2}R^2(\dot{H} + H^2)\delta_{rp}\delta_{sq}\Delta^{pq} \\ & - \frac{1}{6}R^2H\delta_{pq}\dot{\Delta}^{pq}\delta_{rs} - \frac{1}{6}R^2(\dot{H} + H^2)\delta_{pq}\Delta^{pq}\delta_{rs} \\ & - \frac{1}{12H}(\delta_{rp}\dot{\Omega}^p{}_{,s} + \delta_{sp}\dot{\Omega}^p{}_{,r}) + \frac{1}{18H}\dot{\Omega}^p{}_{,p}\delta_{rs} \\ & + \frac{1}{12H^2}(\dot{H} + H^2)(\delta_{rp}\Omega^p{}_{,s} + \delta_{sp}\Omega^p{}_{,r}) - \frac{1}{18H^2}(\dot{H} + H^2)\Omega^p{}_{,p}\delta_{rs} \\ & - \frac{1}{2}\Omega_{,rs} + \frac{1}{6}\delta^{pq}\Omega_{,pq}\delta_{rs} - \frac{1}{2}\delta_{pq}\delta_{rn}\delta_{sm}S^{pmnq} \\ & + \frac{1}{6}\delta_{pq}\delta_{nm}\delta_{rs}S^{pmnq}\end{aligned} \tag{5.12b}$$

As regards $\delta H_{\alpha\beta}$, the treatment of this quantity, while certainly possible, is formally too elaborate for the present work.

Using equations (5.6) and (5.7), we get explicit expressions for Ω^r and $\Gamma_{,r}$ in terms of δX^r and δY^r, which describe the density and temperature inhomogeneities we wish to investigate. The quantity $X^\alpha \cong \epsilon\, \delta X^\alpha$ is measurable in the sense that, as observed already by Ellis and Bruni (1989), "(*a*) it can be determined from virial theorem estimates (indeed, dynamical mass estimates determine precisely spatial density gradients), and (*b*) the contribution to it from luminous matter can be found by observing gradients in the number of observed sources and estimating the mass-to-light ratio [Kristian and Sachs (1966), equation (39)]." Further, for small ϵ, the magnitude of $\epsilon\, \delta Y^r$ directly indicates how rapid the spatial variation of temperature is. Thus this quantity seems to be measurable as well. From equations (5.10) it follows in turn that $\Delta^{\alpha\beta}$ is uniquely determined by $\delta\sigma_{\alpha\beta}$ when R and H are known. Finally, in these calculations, one sees illustrated why it is that basic gauge-invariant variables play such a large part in perturbation theory. They remind one that any gauge-invariant quantity is obtainable directly from D $:= \{\chi, \Gamma, \Omega, \Omega^r, \Delta, \Delta^{rs}, S^{ijrs}\}$ through purely algebraic and differential operations. At first sight, this fact seems highly mysterious in the case of equation (5.12b), for how could the infinitesimal changes in $E_{\alpha\beta}$ relate to D if these changes depend on the metric tensor perturbations $Q^{\alpha\beta}$ alone? However, equation (5.12b) only illustrates a general theorem first established by Banach and Piekarski (1996a, Appendix) for an almost-Robertson–Walker universe model.

Ellis and Bruni (1989) have developed a formalism based on gauge-invariant variables which are perturbations of quantities which vanish on the background. The main purpose of this discussion was to show that all these variables can be derived in a straightforward manner from D. Our analysis here is not complete, of course, but this paper does not allow space for the more detailed comparisons. Nevertheless, in Section 6 we briefly describe how our basic variables relate to those obtained by Bardeen (1980).

5.2. Relationship to Thermodynamics

A perfect fluid usually satisfies the property that

$$(nsu^\alpha)_{;\alpha} = 0 \tag{5.13}$$

where s is the specific entropy, i.e., the entropy per particle. Another useful quantity is the specific free energy

$$\psi = \frac{e}{n} - Ts \tag{5.14}$$

It is convenient to take n and T as independent quantities and to postulate an equation of state

$$\psi = \psi(n, T) \tag{5.15}$$

The first law of thermodynamics (Smarr and Taubes, 1980),

$$d\psi = n^{-2}p\, dn - s\, dT \tag{5.16}$$

then defines p and s in terms of n and T:

$$p = n^2 \frac{\partial \psi}{\partial n}, \qquad s = -\frac{\partial \psi}{\partial T} \tag{5.17}$$

If we put $s = -\partial\psi/\partial T$ in equation (5.14), we find that

$$e = n\left(\psi - T\frac{\partial \psi}{\partial T}\right) \tag{5.18}$$

Thus, so long as the fluid remains in local equilibrium, the values of p and e can be ascertained from information which is static and universal. This information consists in the functional form of the relation (5.15).

When equation (5.13) is combined with the equation of balance of number density,

$$(nu^{\alpha})_{;\alpha} = 0 \tag{5.19}$$

we derive that the perfect fluid is locally adiabatic:

$$u^{\alpha} s_{,\alpha} = 0 \tag{5.20}$$

That is, entropy is constant along the flow lines of the fluid. In this way, we arrive at the following conclusion: the specific entropy s is a scalar that is constant in the unperturbed space-time. More suggestively, directly from equation (5.20) we see that the value of the entropy perturbation δs is invariant under gauge transformations. Differentiation of $s = -\partial\psi/\partial T$ with respect to ϵ at $\epsilon = 0$ then will yield the additional gauge-invariant quantity we seek. In fact, we find from equations (5.17), (5.18), and

$$\dot{T}_0 = -\frac{3}{e_T}(e_0 - n_0 e_M + p_0)H = -3T_0 \frac{p_T}{e_T} H \tag{5.21}$$

that

$$\delta s = \frac{e_T}{n_0}\Gamma \tag{5.22}$$

where Γ is given by equation (3.4b). Thus Γ is proportional to the entropy perturbation δs.

Of course, the interpretation just presented for the gauge-invariant quantity Γ is of a quality essentially different from those of the previous sections, which involved no arguments based on the "first" and "second" laws of thermodynamics.

6. DISCUSSION AND CONCLUSION

In this and a companion paper (Banach and Piekarski, 1996a), we have presented a totally gauge-invariant framework for studying the time development of perturbations away from homogeneous, isotropic cosmological models permeated by a nonbarotropic perfect fluid. Nonbarotropic perfect fluids are perfect fluids where (e, p) and (n, T) are functionally dependent: $e = e(n, T)$, $p = p(n, T)$. Matter, which locally is subject to energy dissipation and hence to viscous stresses, was not discussed, however, because (in all probability) we shall gain more by working on kinetic theory (Ellis *et al.*, 1983; Banach and Piekarski, 1994a–c; Banach and Makaruk, 1995) than by trying to develop formal phenomenological theories remotely related to real physical situations. Nevertheless, our approach is sufficiently flexible and broadly based that it can be easily extended to materials more complex than we have considered here.

The general covariance of Einstein's theory of gravity implies that two perturbations $\delta\mathcal{G}_0$ and $\delta\mathcal{G}_0'$ are equivalent if (and only if) they differ by the action of an "infinitesimal diffeomorphism" on the background solution $\mathcal{G}_0$. The corollary of this observation is as follows: the mathematical object of most physical interest is not just one perturbation $\delta\mathcal{G}_0$, but a whole equivalence class $[\delta\mathcal{G}_0]$ of all perturbations $\delta\mathcal{G}_0'$ which are equivalent to $\delta\mathcal{G}_0$. This class reduces to $\delta\mathcal{G}_0$ itself if and only if for all v, $\mathcal{L}_v\mathcal{G}_0$ can be set equal to zero. Often, so long as the general condition $\mathcal{L}_v\mathcal{G}_0 \neq 0$ holds, one works with just one representative member $\delta\mathcal{G}_0'$ of the equivalence class $[\delta\mathcal{G}_0]$. However, the resulting problem is that the quantity M (the fractional variation in density along a world line) usually determined in perturbation calculations is completely dependent on the gauge chosen and hence *has no physical reality.* For this reason, $\delta\mathcal{G}_0$ does not appear to be the object to use to formulate perturbation theory. The equivalence class $[\delta\mathcal{G}_0]$ substitutes for $\delta\mathcal{G}_0$ for that role.

Nevertheless, it is not simple to achieve the stated aim in perturbation theory by means of the equivalence class $[\delta\mathcal{G}_0]$, since it is not uniquely determined from the standard gauge-invariant variables, yet the very nature of gauge-invariant cosmological perturbations we have in mind demands some sort of analytical expression to be used for the explicit and unique characterization of $[\delta\mathcal{G}_0]$. In the past, comparatively little attention has been focused upon this general problem. With the help of a geometric approach to cosmological perturbations (Banach and Piekarski, 1996a), we have found a complete set D of basic gauge-invariant variables that uniquely represent

$[\delta\mathscr{G}_0]$ in almost-Robertson–Walker universe models. Also, by appropriate combinations and differentiations of these basic variables we have proved that any complicated gauge-invariant quantity can be expressed in terms of $\mathbb{D}$. These results were derived without making any explicit or implicit reference to the method of scalar-, vector-, and tensor-field harmonics. In fact, the analysis below shows that this method, which originally was proposed to effectively separate out the time and space variations (Gerlach and Sengupta, 1978a), is very useful but not necessary for giving a complete gauge-invariant formulation of perturbation theory. Here it can be used to decompose equations (3.6) harmonically. We refer the interested reader to Kalnins and Miller (1991) for a detailed discussion of this point.

Only after one has considered these sorts of basic questions will one be able to focus upon such comparatively practical matters as studying the time development of $\mathbb{D}$ or determining a set of gauge-invariant quantities that are directly related to density fluctuations [for a summary of earlier approaches to this problem see, e.g., Mukhanov *et al.* (1992) and the literature quoted there]. Thus, it was shown in Section 3.2 that, in a perfect-fluid approximation, the basic variables satisfy equations (3.6), partial differential equations involving Γ, Ω, Ω^r, Δ, Δ^{rs}, and S^{ijrs} in tractable combinations. If, furthermore, one is willing to model the pressure effects in an otherwise pressure-free background ideal fluid, proceeding as if the system in question were a rarefied gas during the dust-dominated epoch, one is led to equations (4.5) that were formulated in Section 4.1. What remains, of course, is to discuss in some greater detail the physical meaning of $\mathbb{D}$; this was done in Section 5.

In particular, the important physical object is the orthogonal density gradient X^r obtained by multiplying Ω^r by $\epsilon n_0 R^{-2}$:

$$X^r \cong \epsilon n_0 R^{-2} \Omega^r \tag{6.1}$$

Ellis and Bruni (1989) gave the first systematic treatment of the properties of X^r. There is a problem with X^r: it is not dimensionless. However, if we define $\overline{X}_r$ by

$$\overline{X}_r := R^2(n_0 H)^{-1}\delta_{rs} X^s \cong \epsilon H^{-1}\delta_{rs}\Omega^s \tag{6.2}$$

we obtain a quantity which is gauge invariant and dimensionless and which in addition embodies most closely the intention of the usual (gauge-dependent) definition M.

How do our variables relate to the gauge-invariant variables ϵ_m and ϵ_g of Bardeen (1980)? [See equations (3.13) and (3.14) in his paper.] A completely general perturbation of the gravitational field can be written as a linear combination of perturbations associated with individual harmonics. Scalar harmonics $Q^{(0)}$ are solutions of the scalar Helmholtz equation (Bardeen, 1980)

$$\delta^{rs} Q^{(0)}{}_{,rs} + k^2 Q^{(0)} = 0 \tag{6.3}$$

The wave number k sets the spatial scale of the perturbation relative to the comoving background coordinates. From now on we assume that the separation of Γ, Ω^r, and Δ^{rs} into individual harmonics has been made. We restrict attention to the so-called scalar perturbations (Mukhanov *et al.*, 1992, p. 212). Let

$$\Pi^r := \frac{1}{3R^2H^2}(-2\dot{H}\Omega^r + T_0 e_T \delta^{rs}\Gamma_{,s}) \tag{6.4a}$$

$$\Pi^{rs} := \frac{1}{2R^2}(\delta^{rp}\Pi^s{}_{,p} + \delta^{sp}\Pi^r{}_{,p}) - \frac{1}{3R^2}\Pi^p{}_{,p}\delta^{rs} + 2R^{-2}\dot{H}\Delta^{rs} \tag{6.4b}$$

Interpreting Π^r in the linear approximation, this quantity can be identified with the fractional energy gradient [see equation (22) in Ellis and Bruni (1989)]:

$$\Pi^\alpha := \frac{1}{e_0}\left[\frac{\partial}{\partial\epsilon}(h^{\alpha\beta}e_{,\beta})\right]_{|\epsilon=0} \tag{6.5}$$

It then follows from the definitions of Π^r and Π^{rs} that

$$\epsilon\Pi^r = -kR^{-2}\,\epsilon_m\delta^{rs}Q_s^{(0)} \tag{6.6a}$$

and

$$\epsilon\Pi^{rs} = k^2R^{-4}\,\epsilon_g\delta^{rp}\delta^{sq}Q_{pq}^{(0)} \tag{6.6b}$$

To obtain these results, we have introduced the vector (Bardeen, 1980)

$$Q_r^{(0)} := -(1/k)Q^{(0)}{}_{,r} \tag{6.7a}$$

and the traceless, symmetric, second-rank tensor (Bardeen, 1980)

$$Q_{rs}^{(0)} := k^{-2}Q^{(0)}{}_{,rs} + \tfrac{1}{3}\delta_{rs}Q^{(0)} \tag{6.7b}$$

Using equations (6.4), we now see that ϵ_m and ϵ_g give us heavily disguised information about Γ, Ω^r, and Δ^{rs}. However, the behavior of ϵ_m and ϵ_g for $k \cong 0$ presents an obstacle from the standpoint of applications; ϵ_m and ϵ_g appear to be singular near $k = 0$. Our approach is to deal directly with D, so that equations (3.6) *remain valid and nontrivial even if* $k = 0$. Thus some aspects of the general method can still be illustrated by assuming that the perturbed metric is also of Robertson–Walker form (Banach and Piekarski, 1994a–c).

As the final point of this discussion, we observe the following: In the case of scalar perturbations (and only in this case), it is interesting to define a potential Φ for δE_{rs}. This can be done by using a formula of the form

$$\delta E_{rs} = (\delta^p{}_r\delta^q{}_s - \tfrac{1}{3}\delta_{rs}\delta^{pq})\Phi_{,pq} \tag{6.8}$$

Thus, to first order in the deviations from the background solution, Φ is a potential for the electric part of the Weyl tensor. This potential was originally introduced by Bardeen (1980); in his notation, $\Phi = \Phi_A = -\Phi_H$. The analysis of Mukhanov *et al.* (1992) supports the interpretation of Φ as the relativistic generalization of a Newtonian gravitational potential [see also Jaffe (1994)]. Of course, in a similar fashion, we can define potentials for our basic gauge-invariant variables Ω^r Δ^{rs} (The potentials for Γ, Ω, and Δ are simply Γ, Ω, and Δ, and S^{ijrs}. Then, since Φ is related to D by equations (5.12b) and (6.8), it will be possible to express Φ in terms of these more elementary potentials. This fact also explains why Φ is not a simple gauge-invariant quantity.

To sum up: In this and the companion paper (Banach and Piekarski, 1996b) we have tried to show how, thanks to the existence of a bijection between $\mathcal{P}/\mathcal{P}_0$ and $\mathcal{D}$, the equivalence class $[\delta\mathcal{G}_0]$ can serve as the basis on which to erect a representation of density inhomogeneities in an almost-Robertson–Walker universe model. We believe that the present analysis puts perturbation theory on a firmer mathematical foundation than that by the previous analyses.

NOTE ADDED IN PROOF

Since this paper was completed, we have been able to improve the discussion of Sections 3.2 and 4.2 and to generalize the results by introducing an optimum system of covariant and gauge-invariant propagation equations, leading to more explicit conclusions. A paper presenting these improvements and generalizations is in press (Banach and Piekarski, 1996b).

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